

Economic Inequality from a Statistical Physics Point of View

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T. Di Matteo, J.B. Rosser, G. Semieniuk and Yong Tao

- *European Physical Journal B* **17**, 723 (2000) > >
- *Reviews of Modern Physics* **81**, 1703 (2009)
- Book *Classical Econophysics* (Routledge, 2009)
- *Entropy* **15**, 5565 (2013); *Eur Phys J Spec Topics* **225**, 3313 (2016)
- *Journal of Economic Interaction & Coordination* **14**, 345 (2019) online 2017
- *Journal of Cleaner Production* **264**, 121420 (2020)

Outline:

- Statistical mechanics of money
 - Debt and financial instability
 - Two-class structure of income distribution
 - Global inequality in energy consumption
- INET grant 2013-2015

Boltzmann-Gibbs versus Pareto distribution



Ludwig Boltzmann (1844-1906)

Boltzmann-Gibbs probability distribution $P(\varepsilon) \propto \exp(-\varepsilon/T)$, where ε is energy, and $T = \langle \varepsilon \rangle$ is temperature.



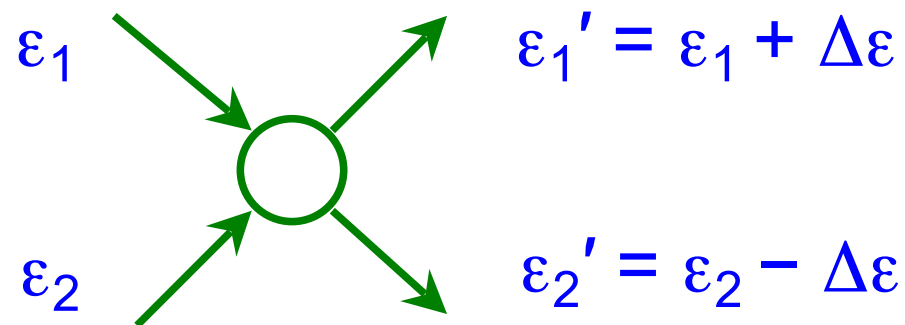
Vilfredo Pareto (1848-1923)

Pareto probability distribution $P(r) \propto 1/r^{(\alpha+1)}$ of income r .

An **analogy** between the distributions of **energy** ε and **money** m or **income** r

Boltzmann-Gibbs probability distribution of **money**

Collisions between atoms



Conservation of energy:

$$\varepsilon_1 + \varepsilon_2 = \varepsilon_1' + \varepsilon_2'$$

Detailed balance:

$$\cancel{w_{12 \rightarrow 1'2'}} P(\varepsilon_1) P(\varepsilon_2) = \cancel{w_{1'2' \rightarrow 12}} P(\varepsilon_1') P(\varepsilon_2')$$

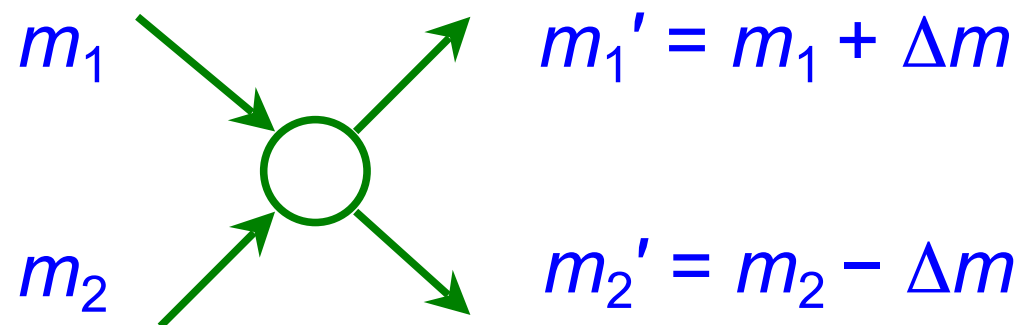
Boltzmann-Gibbs probability distribution $P(\varepsilon) \propto \exp(-\varepsilon/T)$ of energy ε , where $T = \langle \varepsilon \rangle$ is temperature. It is **universal** – independent of model rules, provided the model belongs to the **time-reversal symmetry** class.

Boltzmann-Gibbs distribution **maximizes entropy** $S = -\sum_{\varepsilon} P(\varepsilon) \ln P(\varepsilon)$ under the constraint of conservation law $\sum_{\varepsilon} P(\varepsilon) \varepsilon = \text{const.}$

Economic transactions between agents

Conservation of money:

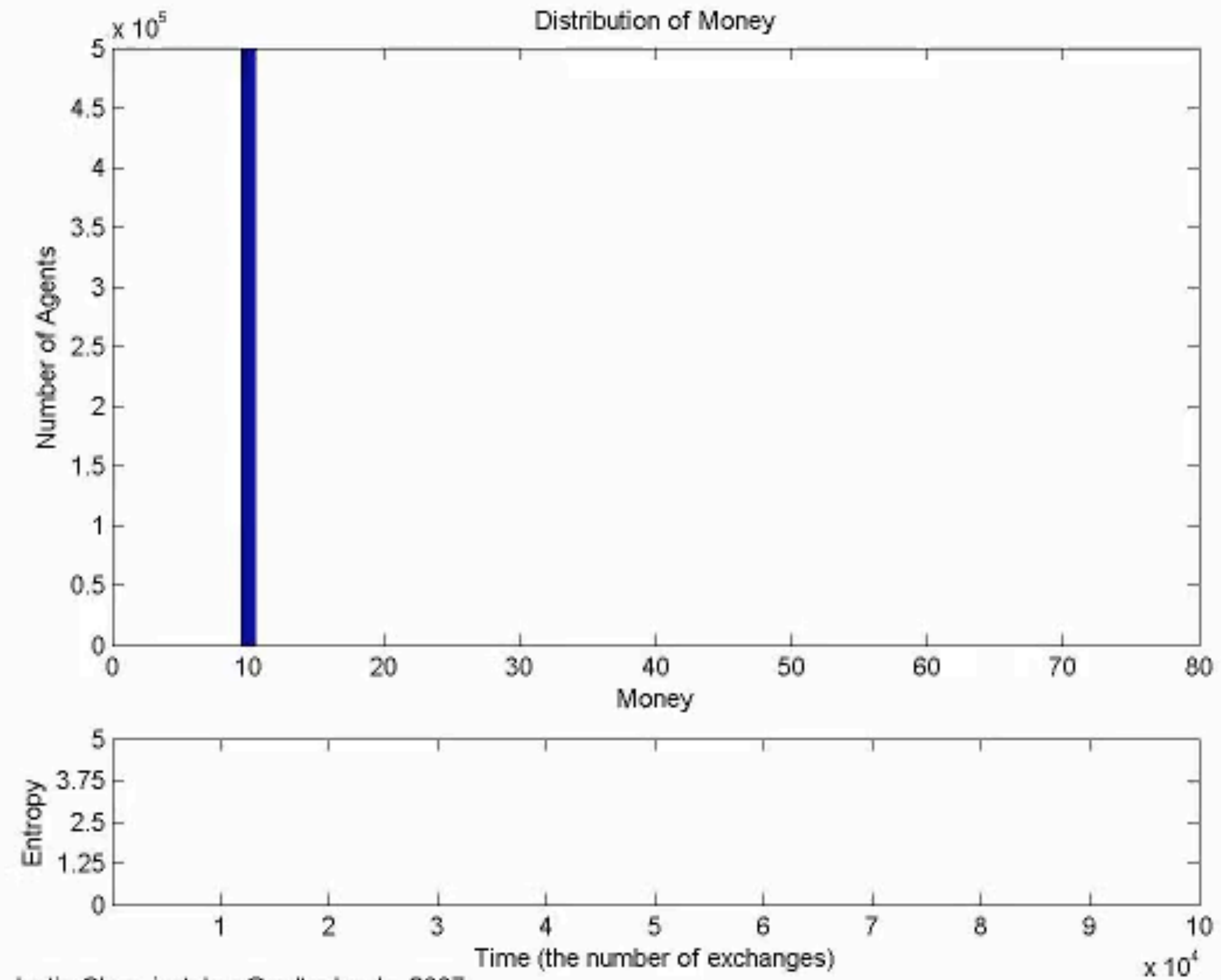
$$m_1 + m_2 = m_1' + m_2'$$



Detailed balance:

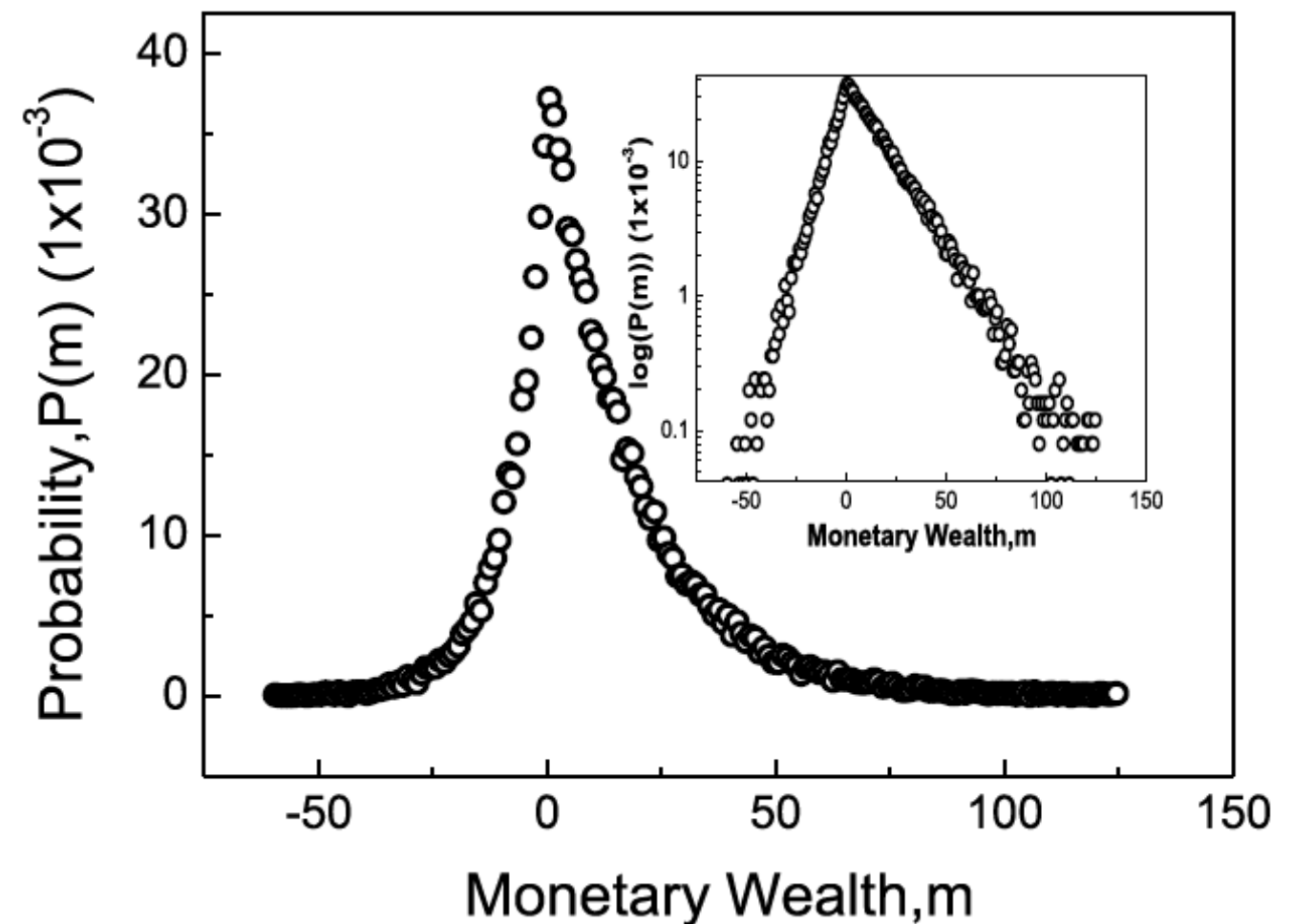
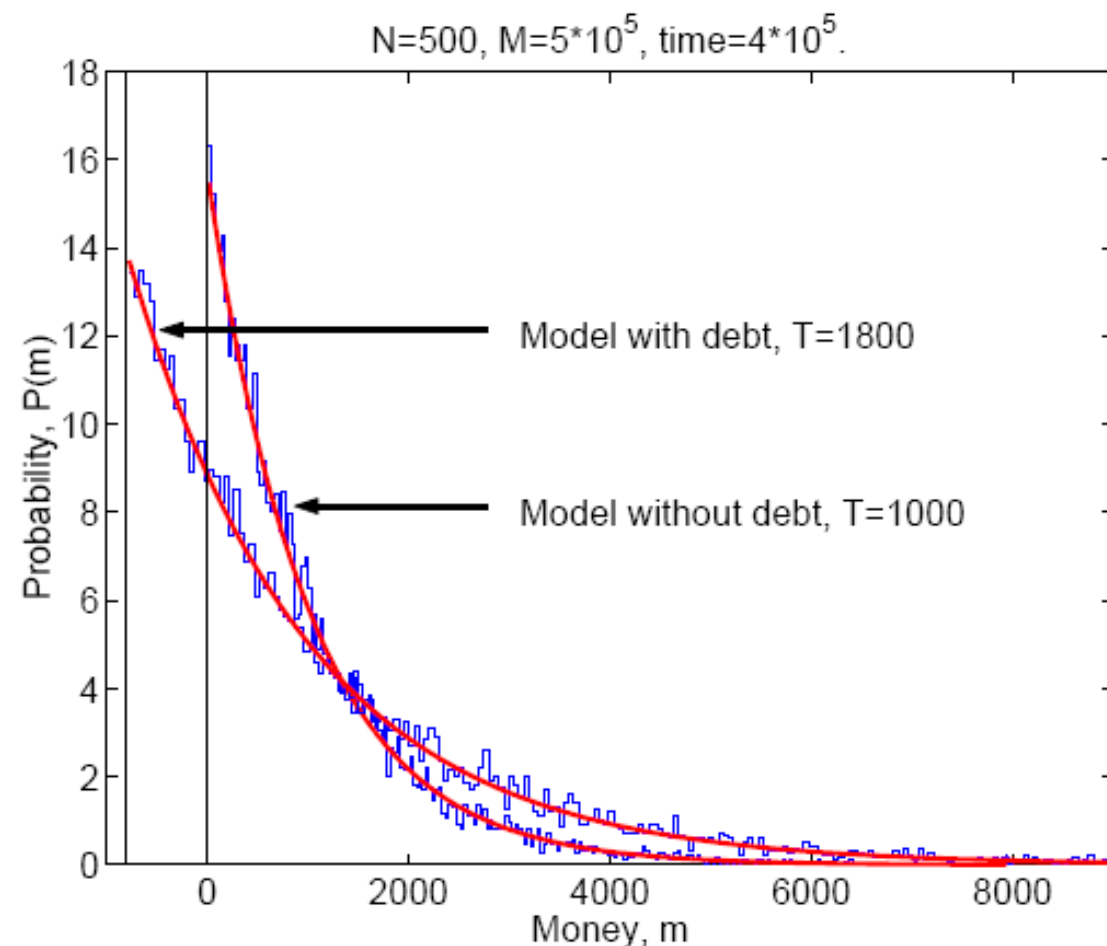
$$w_{12 \rightarrow 1'2'} P(m_1) P(m_2) = w_{1'2' \rightarrow 12} P(m_1') P(m_2')$$

Boltzmann-Gibbs probability distribution $P(m) \propto \exp(-m/T)$ of **money** m , where $T = \langle m \rangle$ is the **money temperature**.



Justin Chen, justchen@caltech.edu, 2007

Money distribution with debt



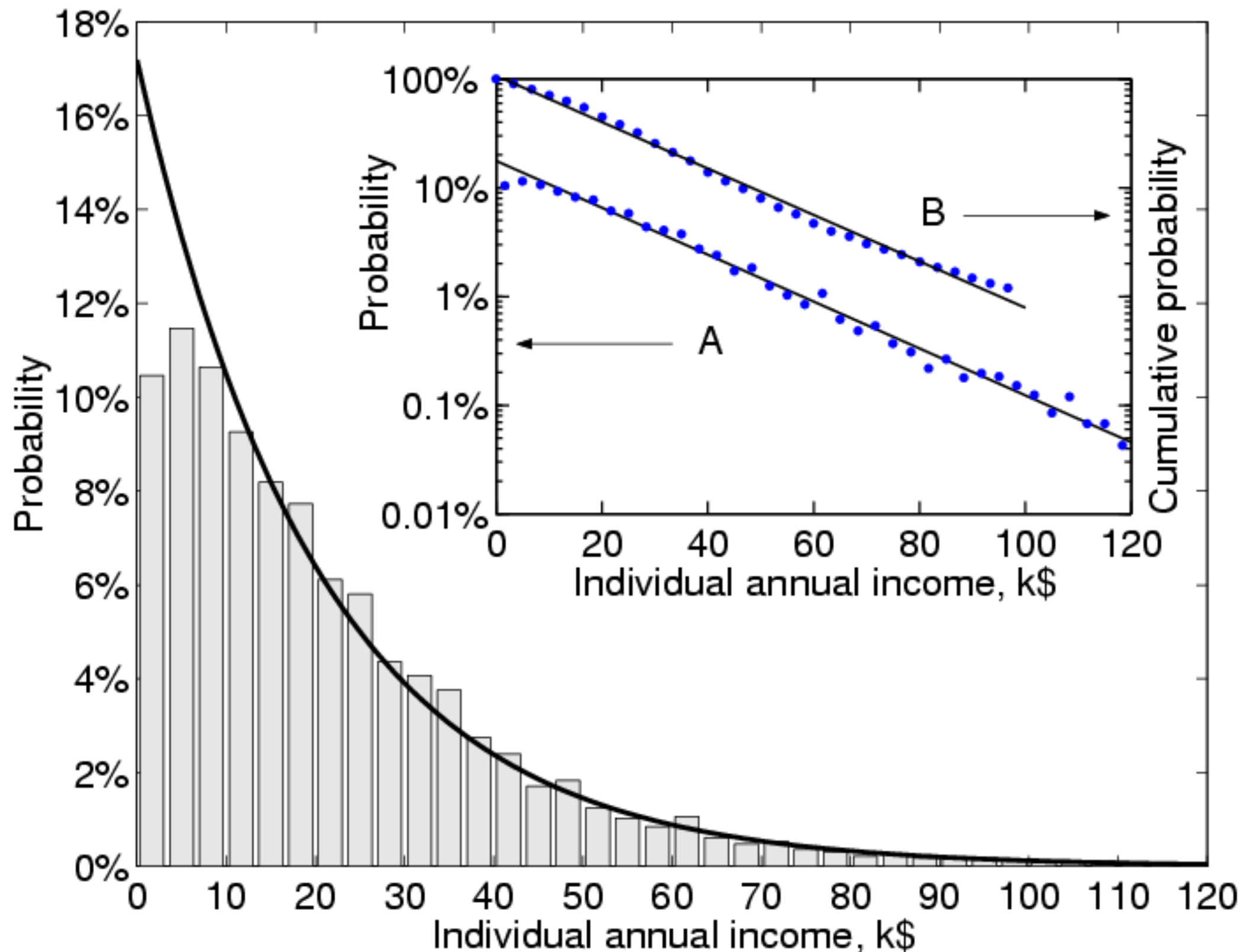
Debt per person is limited to 800 units.

Total debt in the system is limited via the Required Reserve Ratio (RRR):

Xi, Ding, Wang, *Physica A* **357**, 543 (2005)

- In practice, RRR is enforced inconsistently and does not limit total debt.
- Without a constraint on debt, the system does **not have a stationary equilibrium**.
- Free market itself does not have an intrinsic mechanism for limiting debt, and there is **no such thing as the equilibrium debt**.

Probability distribution of individual income

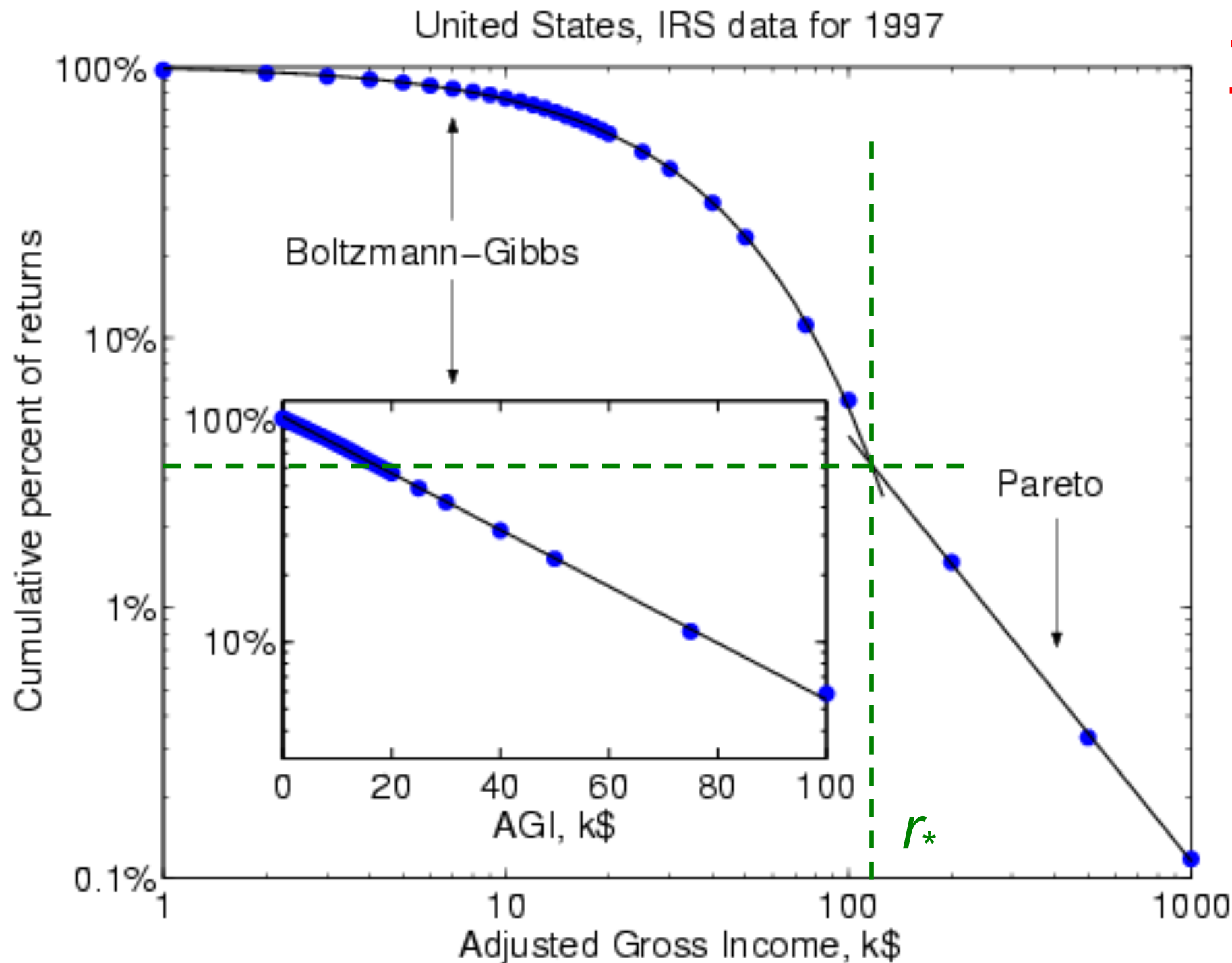


US Census
data 1996 –
histogram and
points A

PSID: Panel
Study of Income
Dynamics, 1992
(U. Michigan) –
points B

Distribution
of income r
is exponential:
 $P(r) \propto e^{-r/T}$

Income distribution in the USA, 1997



Two-class society

Upper Class

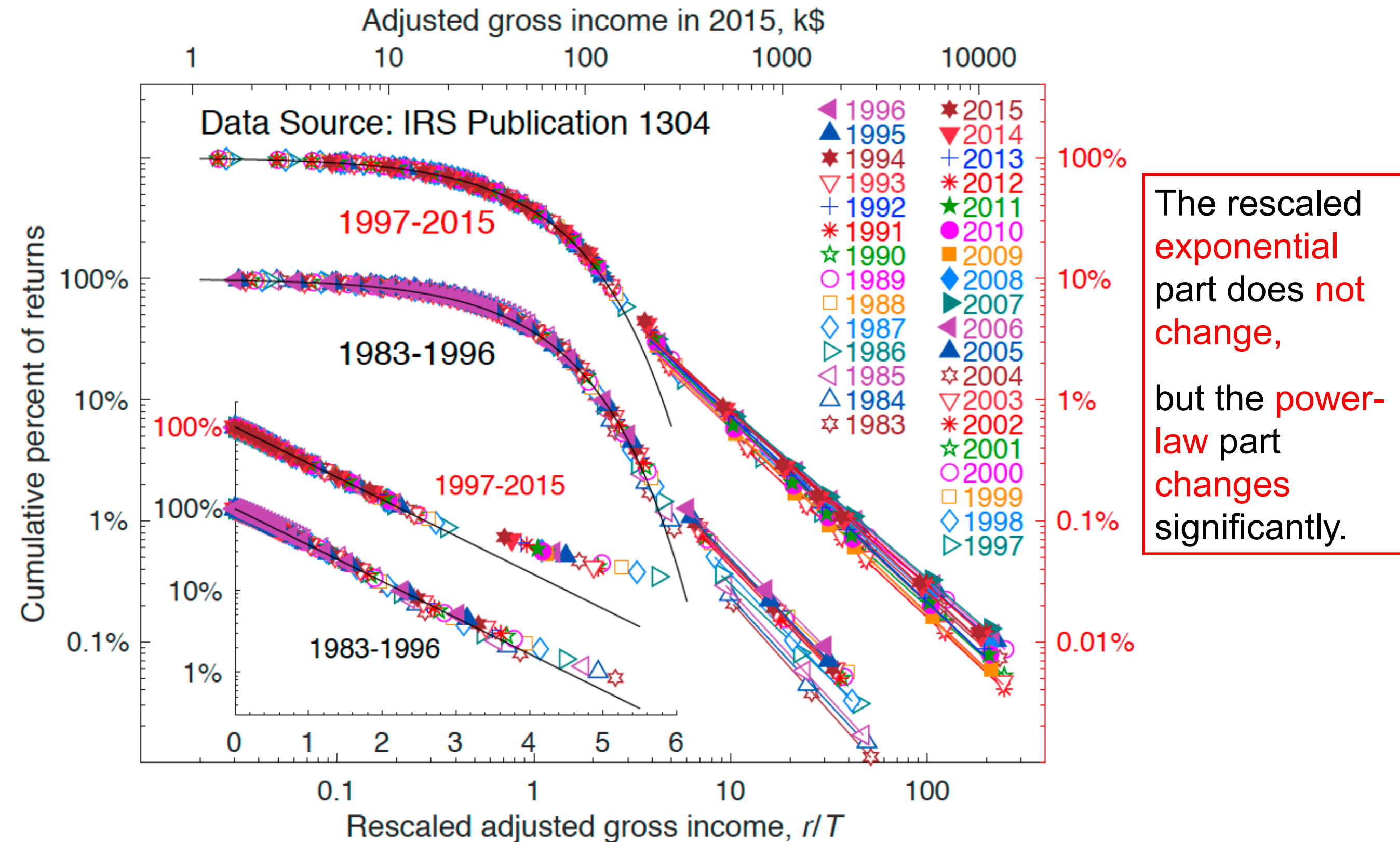
- Pareto power law
- 3% of population
- 16% of income
- Income > 120 k\$: investments, capital

Lower Class

- Boltzmann-Gibbs exponential law
- 97% of population
- 84% of income
- Income < 120 k\$: wages, salaries

“Thermal” bulk and “super-thermal” tail distribution

Income distribution in the USA, 1983-2015



Lorenz curves and income inequality

Lorenz curve ($0 < r < \infty$):

$$x(r) = \int_0^r P(r') dr'$$

$$y(r) = \int_0^r r' P(r') dr' / \langle r' \rangle$$

For exponential distribution, $G=1/2$ and the Lorenz curve is

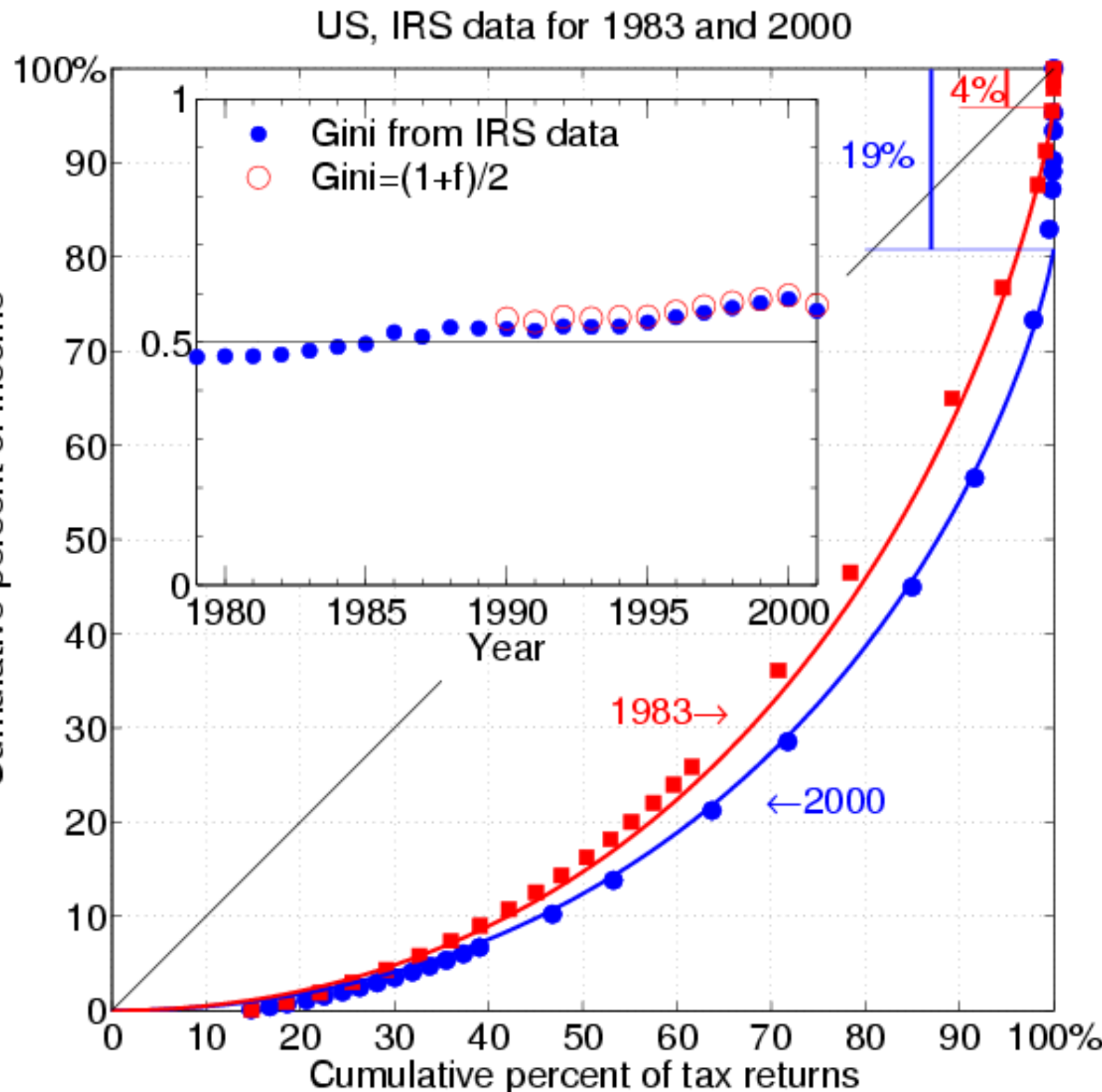
$$y = x + (1-x) \ln(1-x)$$

With a tail, the Lorenz curve is

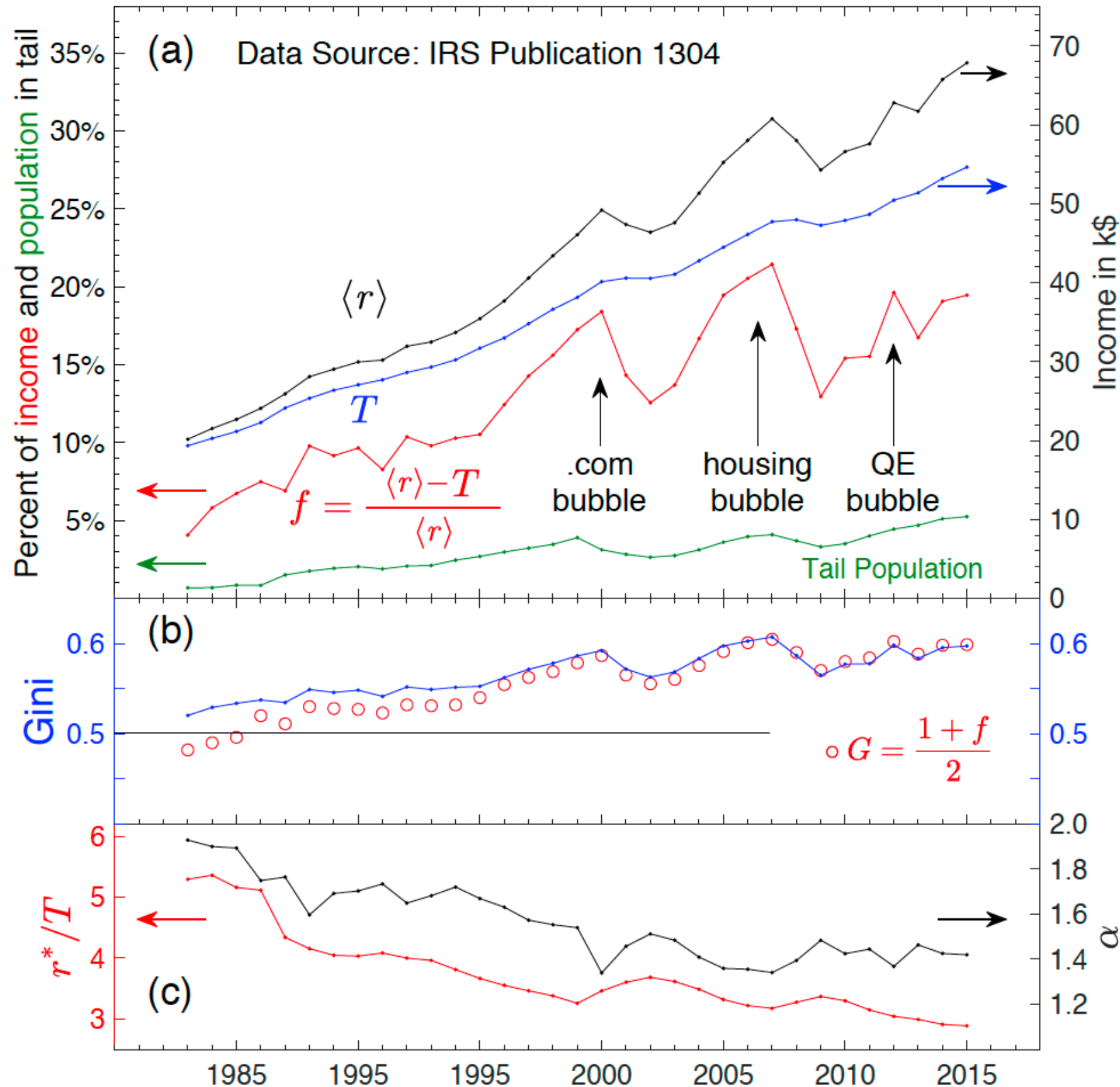
$$y = (1-f)[x + (1-x) \ln(1-x)] + f \Theta(x-1),$$

where f is the tail income, and Gini coefficient is $G=(1+f)/2$.

A measure of inequality, the Gini coefficient is $G = \frac{\text{Area}(\text{diagonal line} - \text{Lorenz curve})}{\text{Area}(\text{Triangle beneath diagonal})}$



Income inequality evolution 1983-2015 in USA



Income inequality
peaks during
speculative bubbles
in financial markets

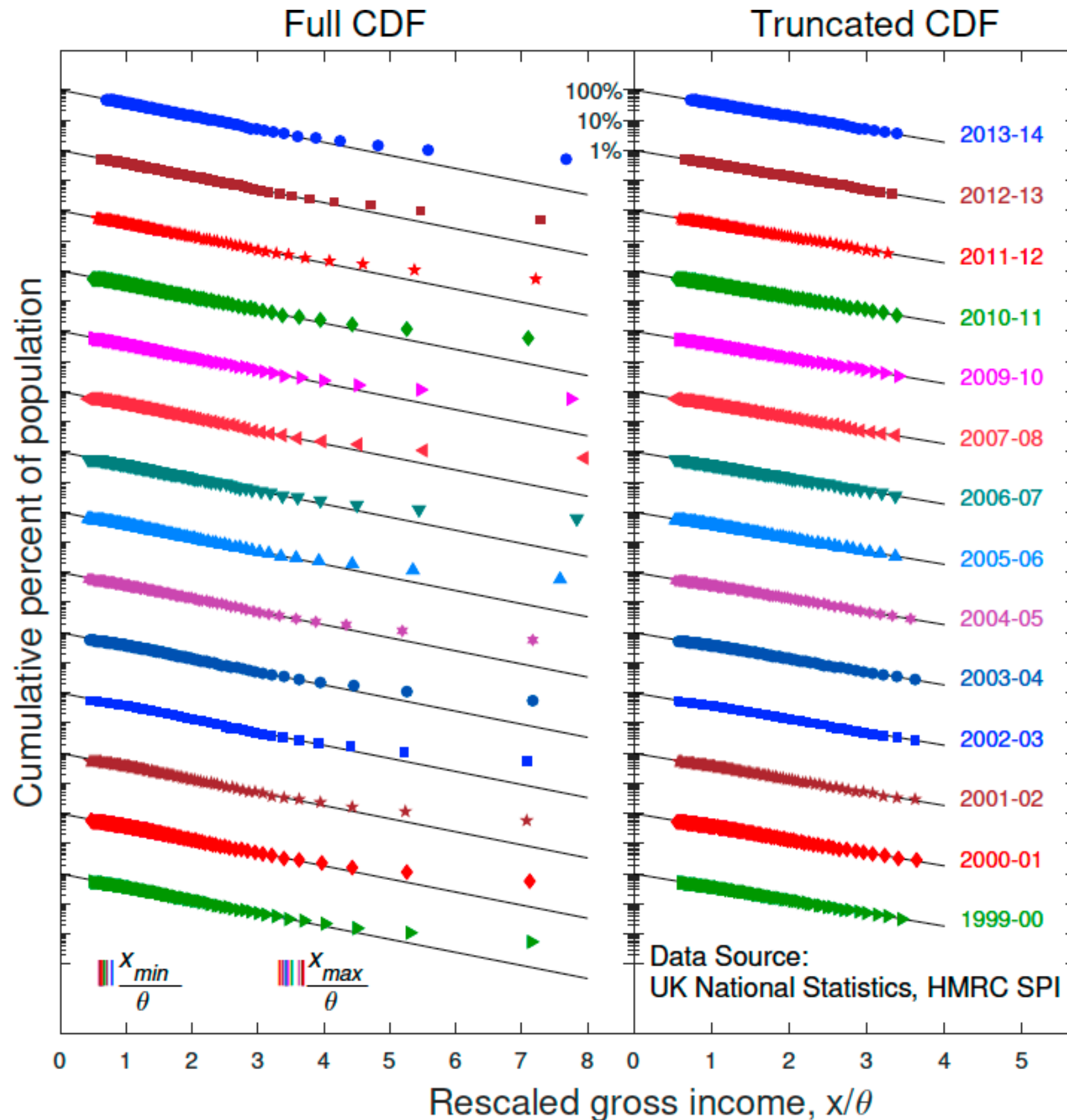
$$f = \frac{\langle r \rangle - T}{\langle r \rangle}$$

f - fraction of
income in the tail

$\langle r \rangle$ - average
income in the
whole system

T - average
income in the
exponential part

Income distribution in the United Kingdom

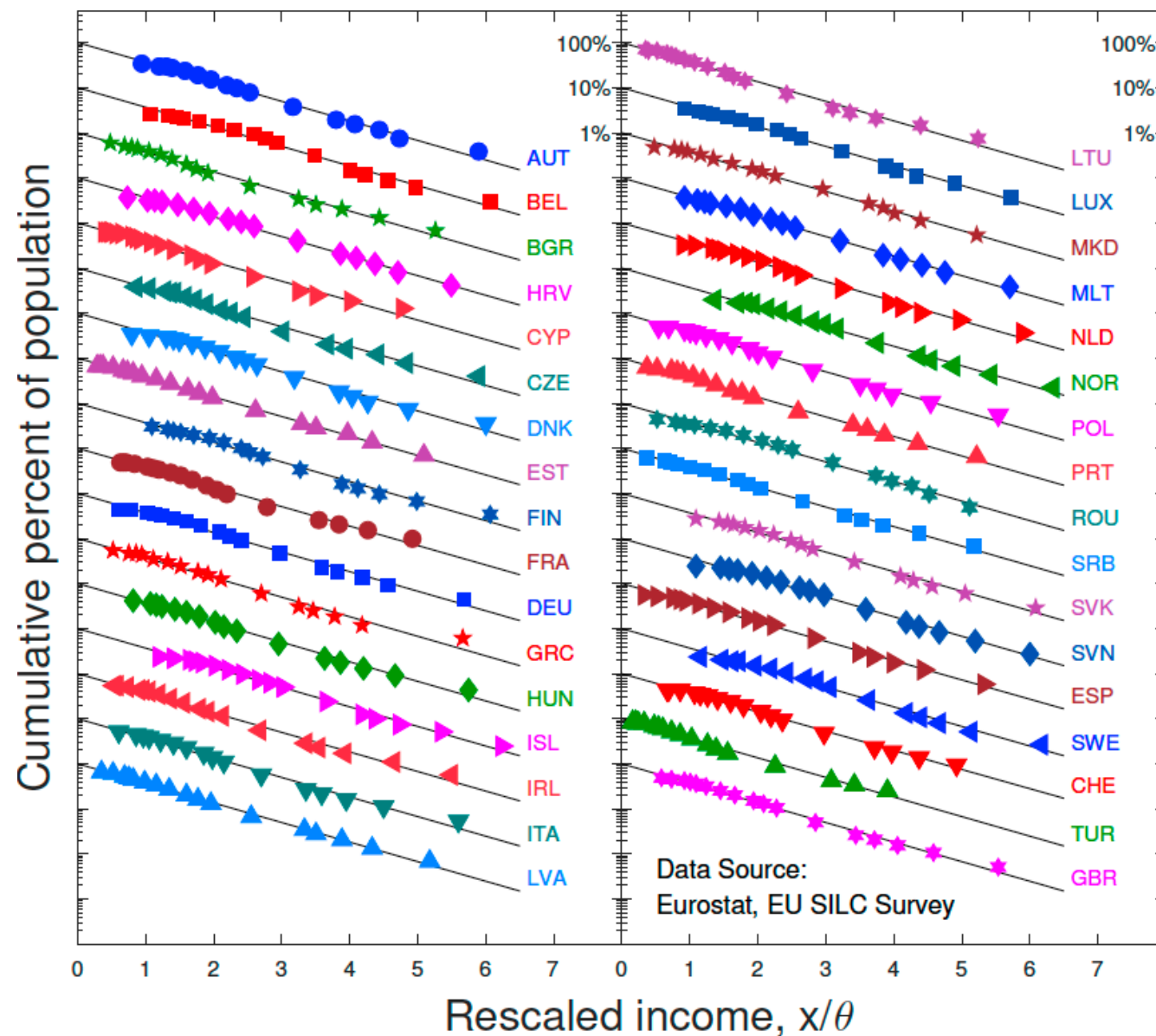


Yong Tao *et al.*,
Exponential structure of
income inequality:
Evidence from 67 countries
*Journal of Economic
Interaction & Coordination*
14, 345 (2019) online 2017

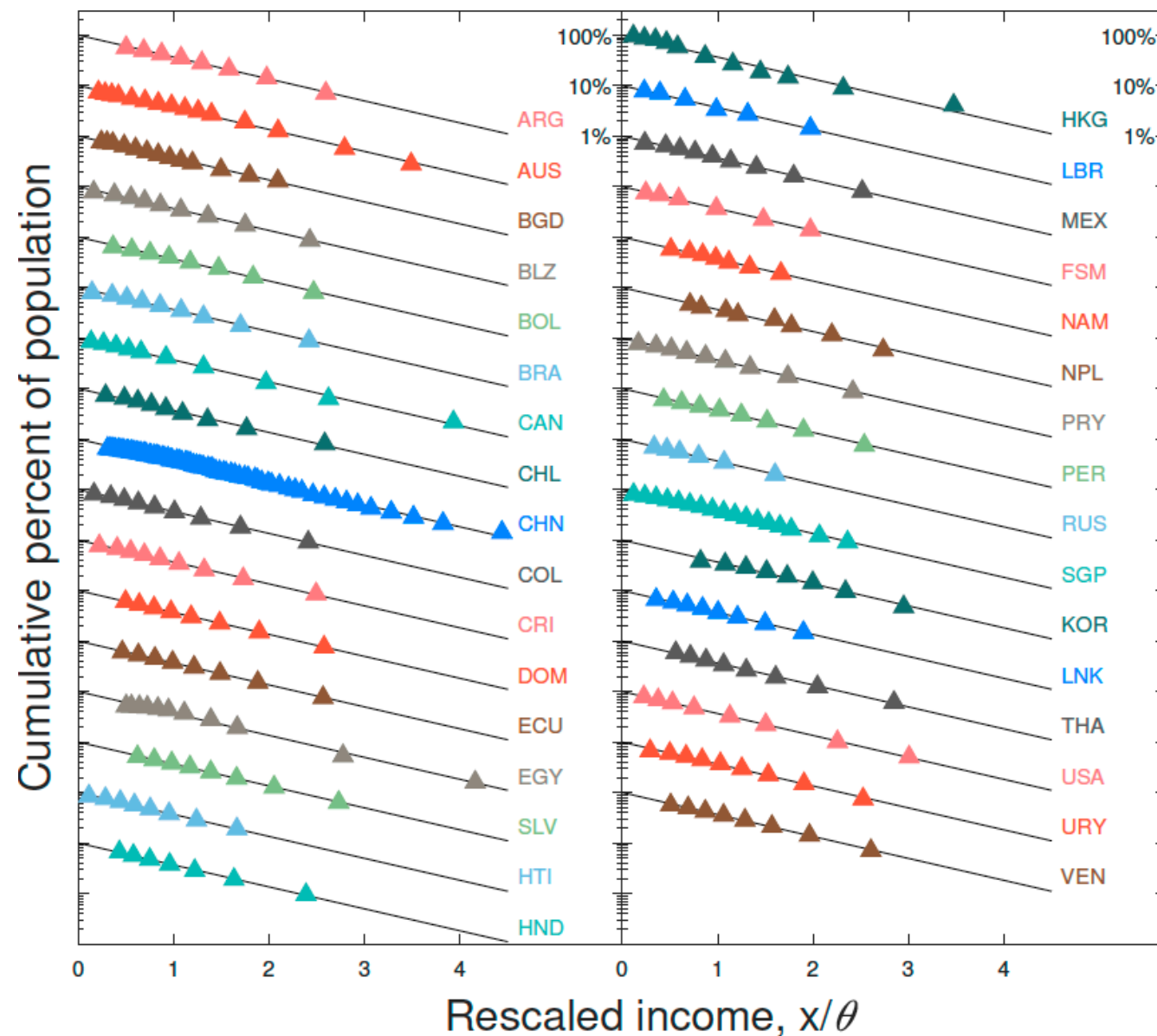
“We have shown that the
standard Arrow–Debreu’s
general equilibrium model
combined with Rawls’
fairness principle naturally
produces the exponential
distribution of income,
which agrees well with the
empirical data for 67
countries around the world”

Exponential income distribution in 67 countries

European Union countries



Other regions of the world

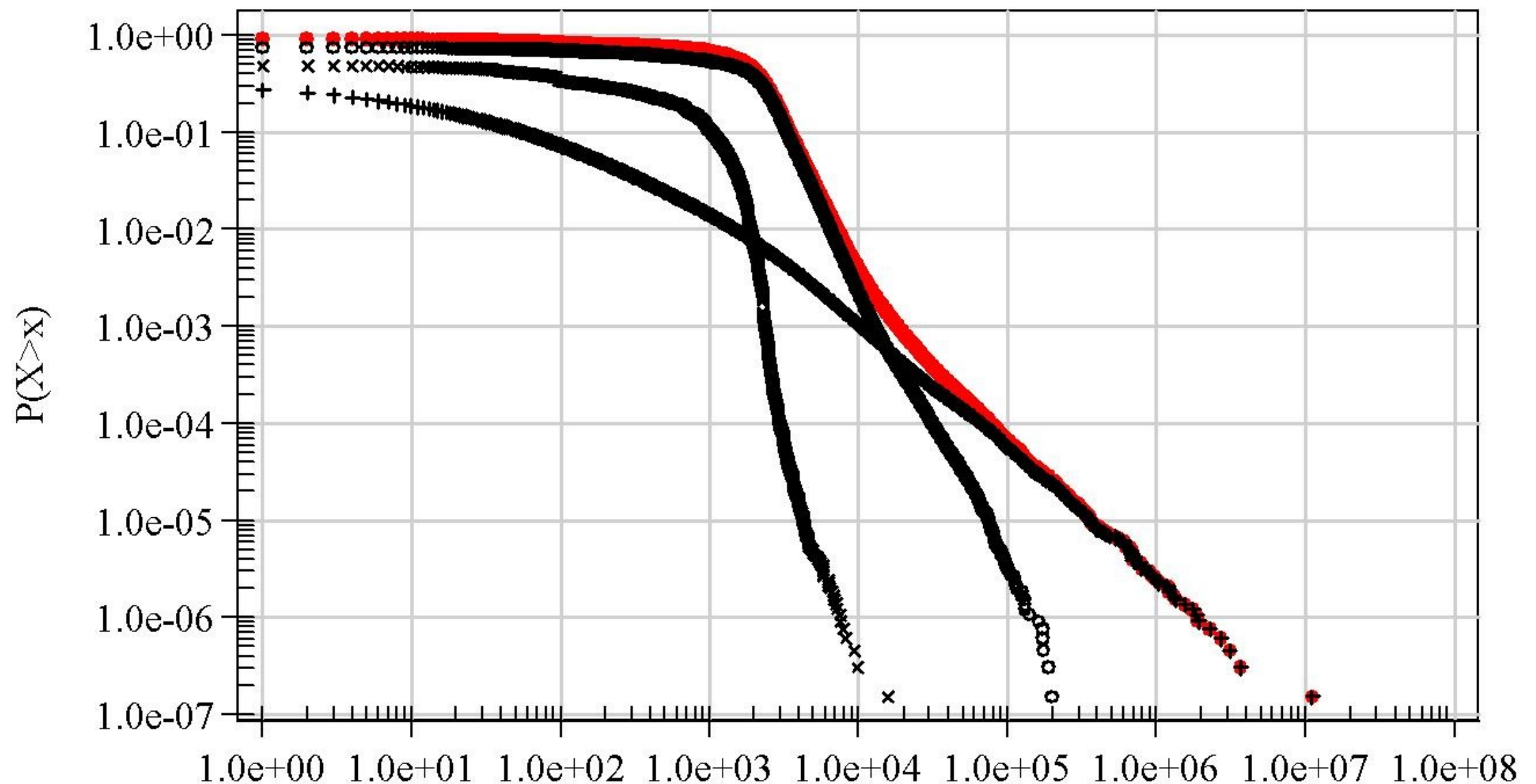


Yong Tao et al., *Journal of Economic Interaction & Coordination* **14**, 345 (2019)

The origin of two classes

- Different sources of income: salaries and wages for the lower class, and capital gains and investments for the upper class.
- Their income dynamics can be described by additive and multiplicative diffusion, correspondingly.
- From the social point of view, these can be the classes of employees and employers, as described by Karl Marx.
- Emergence of classes from the initially equal agents was simulated by Ian Wright “The Social Architecture of Capitalism” *Physica A* 346, 589 (2005), see also the book “Classical Econophysics” (2009)

Income distribution in Sweden



The data plot from
Fredrik Liljeros and Martin Hällsten,
Stockholm University

- Total incomes
- Work
- + Capital
- × Social transfers

Diffusion model for income kinetics

Suppose income changes by small amounts Δr over time Δt .
Then $P(r,t)$ satisfies the **Fokker-Planck equation** for $0 < r < \infty$:

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial r} \left(AP + \frac{\partial}{\partial r} (BP) \right), \quad A = - \left\langle \frac{\Delta r}{\Delta t} \right\rangle, \quad B = \left\langle \frac{(\Delta r)^2}{2\Delta t} \right\rangle.$$

For a stationary distribution, $\partial_t P = 0$ and $\frac{\partial}{\partial r} (BP) = -AP$.

For the **lower class**, Δr are independent of r – **additive diffusion**, so A and B are constants. Then, $P(r) \propto \exp(-r/T)$, where $T = B/A$, – **an exponential distribution**.

For the **upper class**, $\Delta r \propto r$ – **multiplicative diffusion**, so $A = ar$ and $B = br^2$.
Then, $P(r) \propto 1/r^{\alpha+1}$, where $\alpha = 1+a/b$, – **a power-law distribution**.

For the **upper class**, income does change in **percentages**, as shown by **Fujiwara, Souma, Aoyama, Kaizoji, and Aoki (2003)** for the tax data in Japan.
For the **lower class**, the data is not known yet.

Additive and multiplicative income diffusion

If the **additive** and **multiplicative** diffusion processes are present **simultaneously**, then $A = A_0 + ar$ and $B = B_0 + br^2 = b(r_0^2 + r^2)$. The stationary solution of the FP equation is

$$P(r) = \frac{C e^{-\frac{r_0}{T} \arctan\left(\frac{r}{r_0}\right)}}{\left[1 + (r/r_0)^2\right]^{1+a/2b}}$$

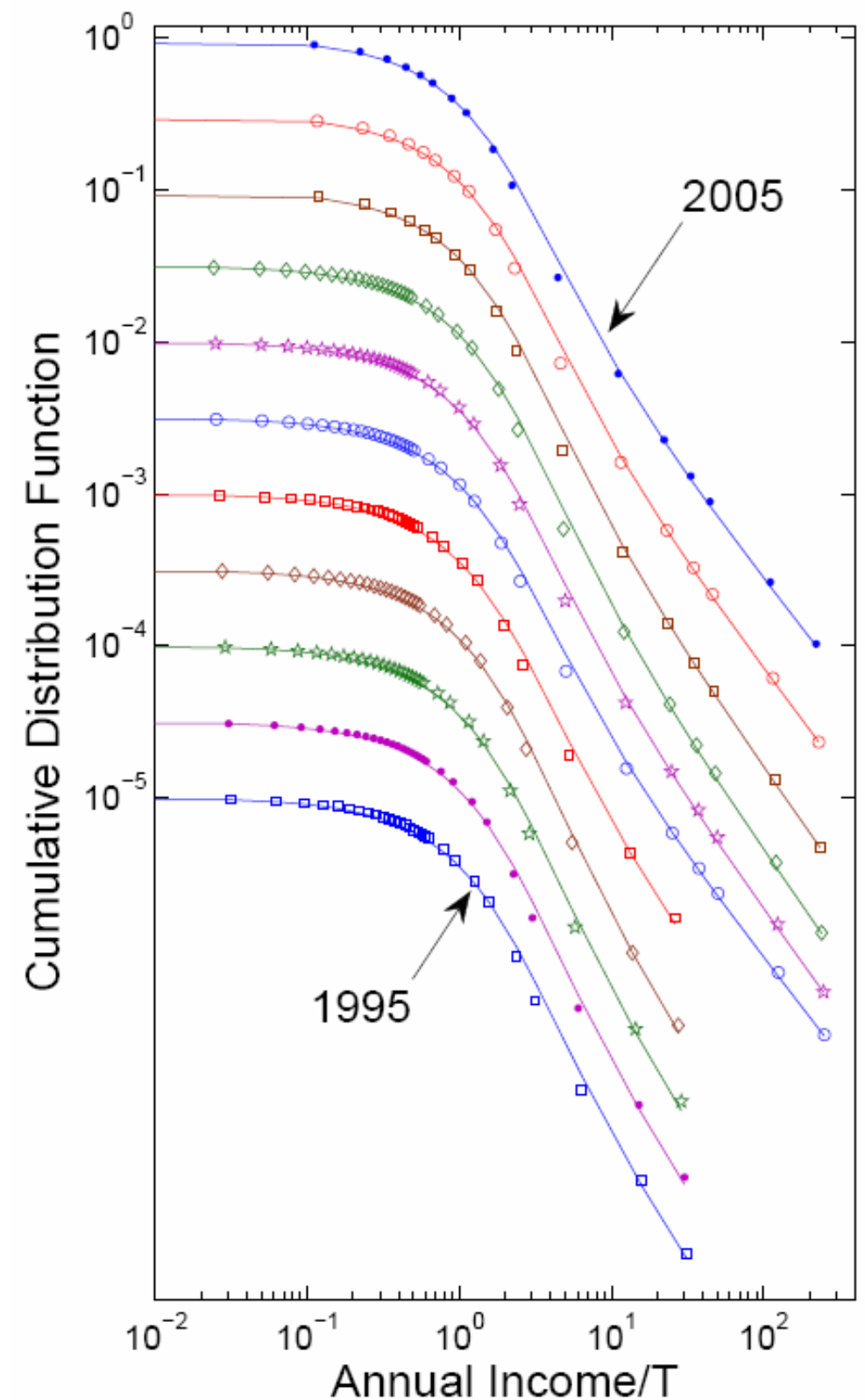
It interpolates between the exponential and the power-law distributions and has 3 parameters:

- $T = B_0/A_0$ – **temperature** of the exponential part
- $\alpha = 1 + a/b$ – **power-law exponent** of the upper tail
- r_0 – **crossover income** between the lower and upper parts.

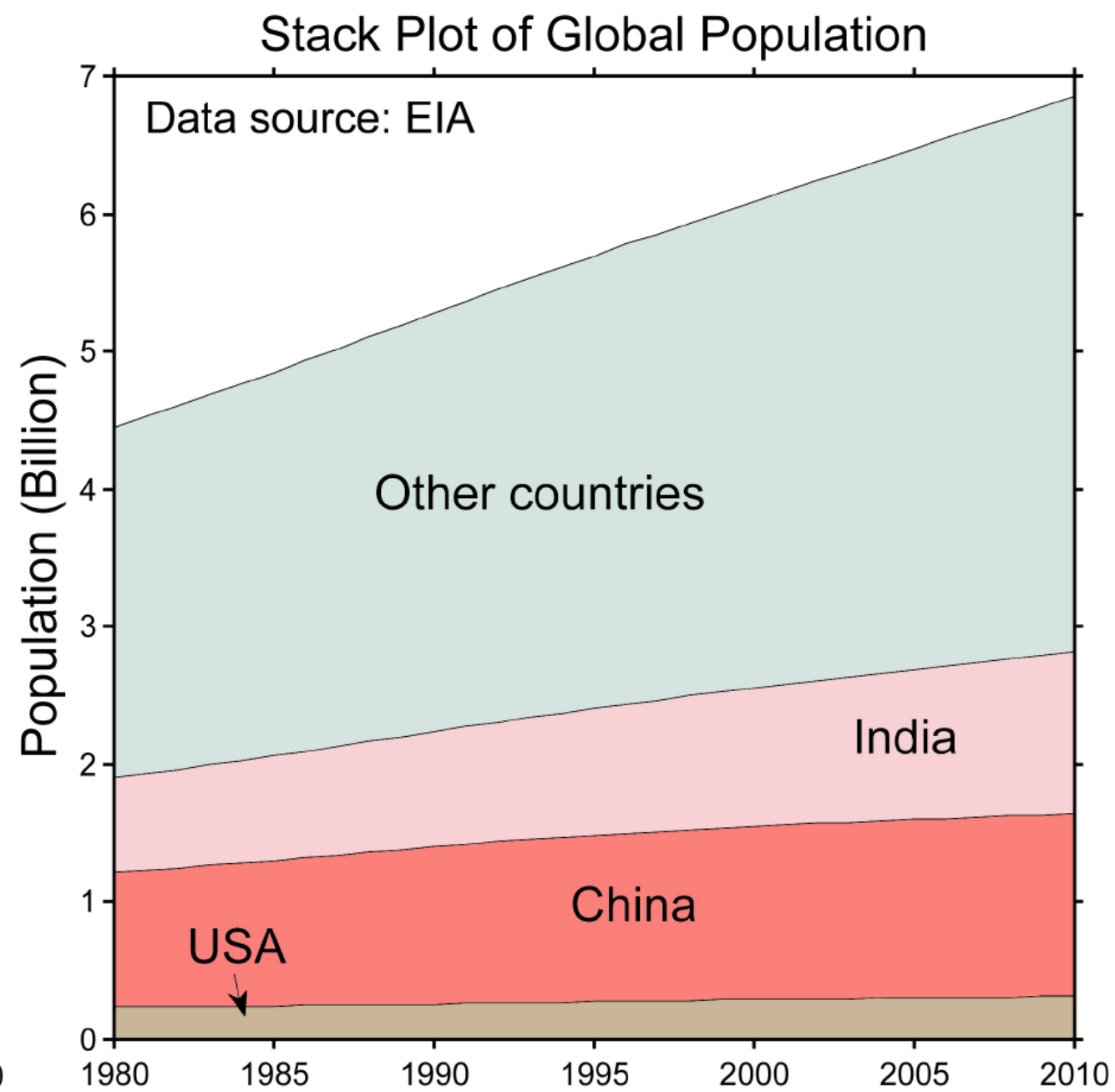
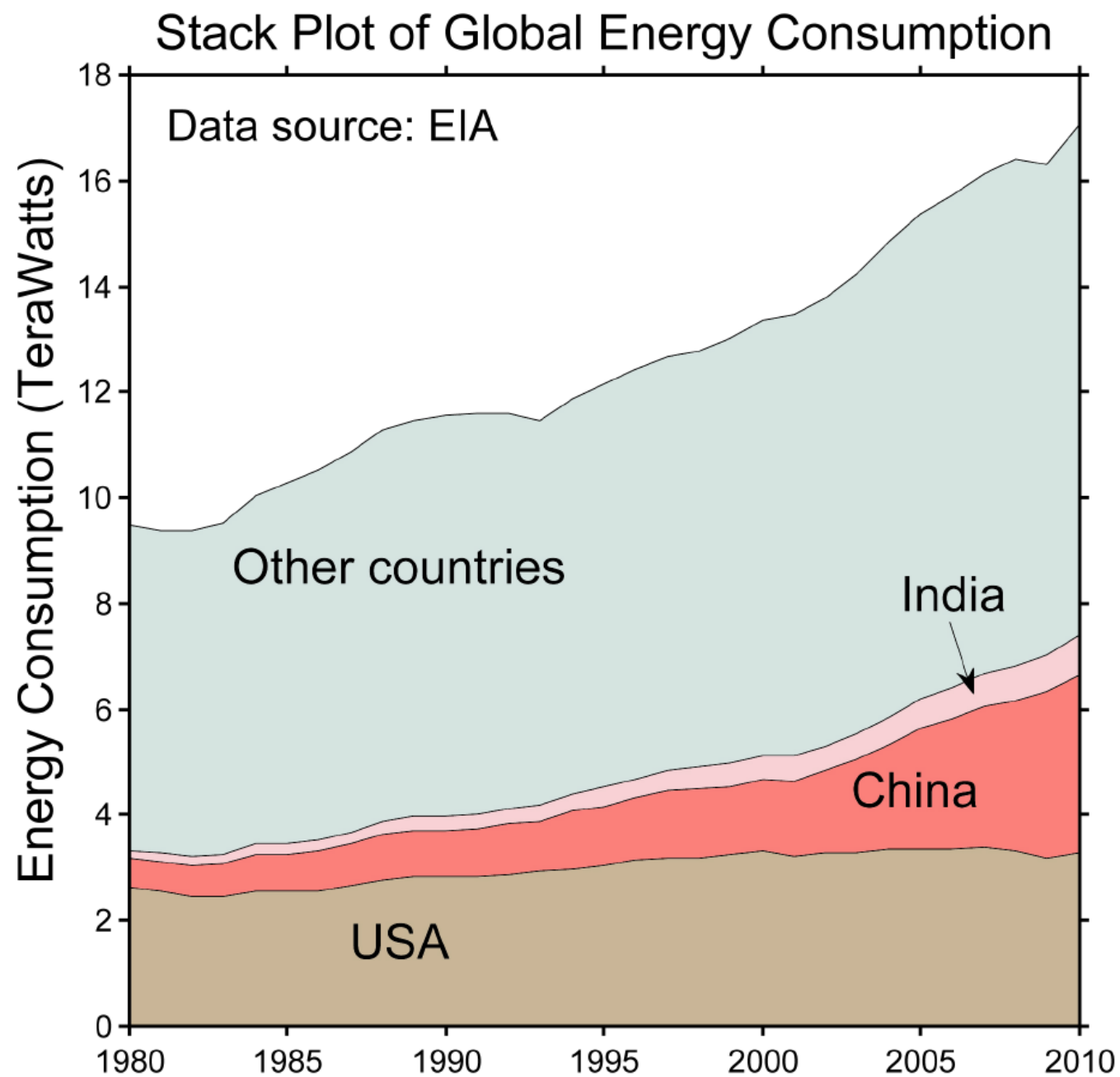
Banerjee & Yakovenko, *RMP* (2009), *NJP* (2010)

Fiaschi & Marsili, *JEBO* (2012)

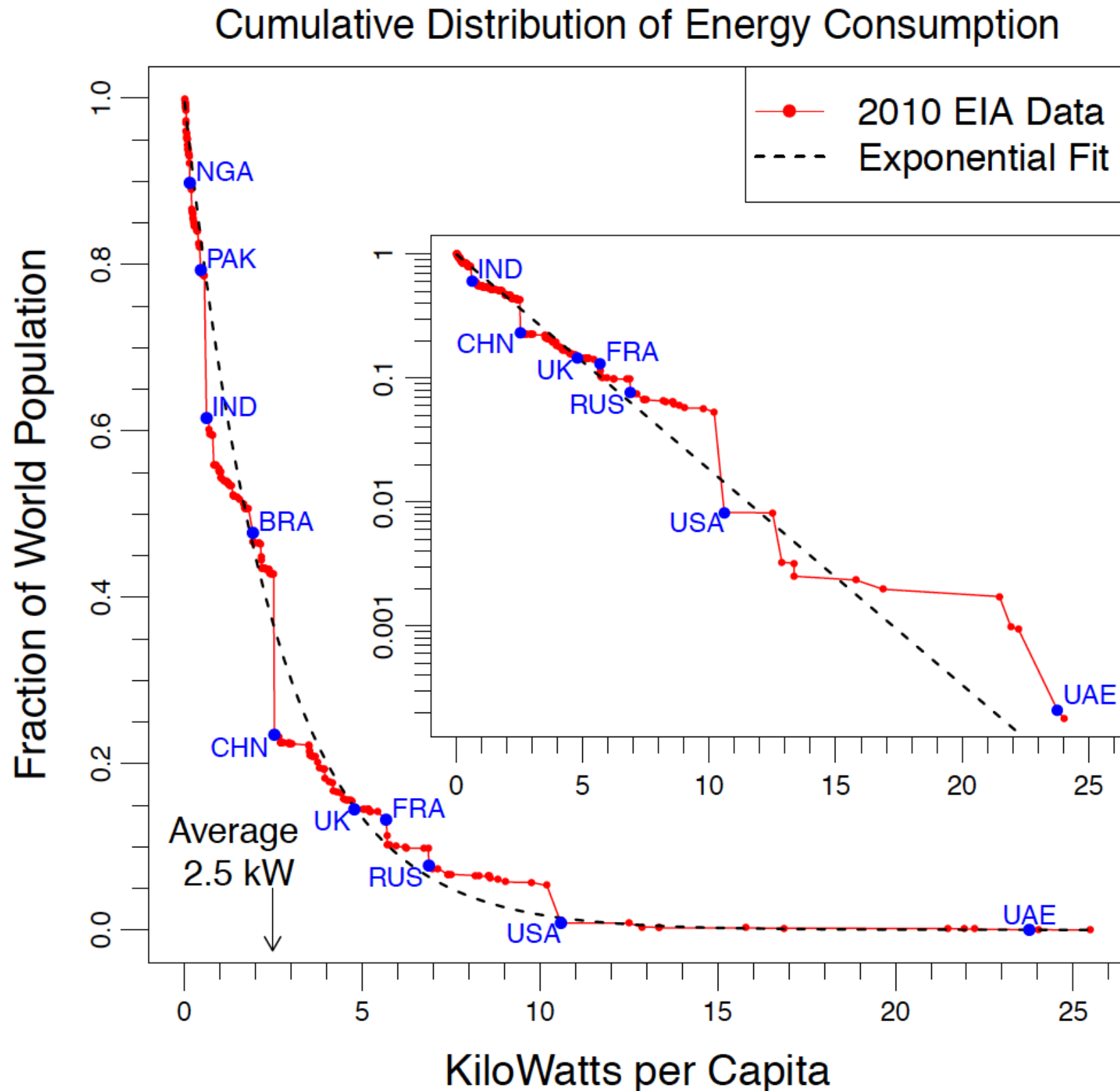
Karl Pearson, *Proc. Roy. Soc. London* (1895)



Energy consumption and population of the biggest countries



Global inequality in energy consumption

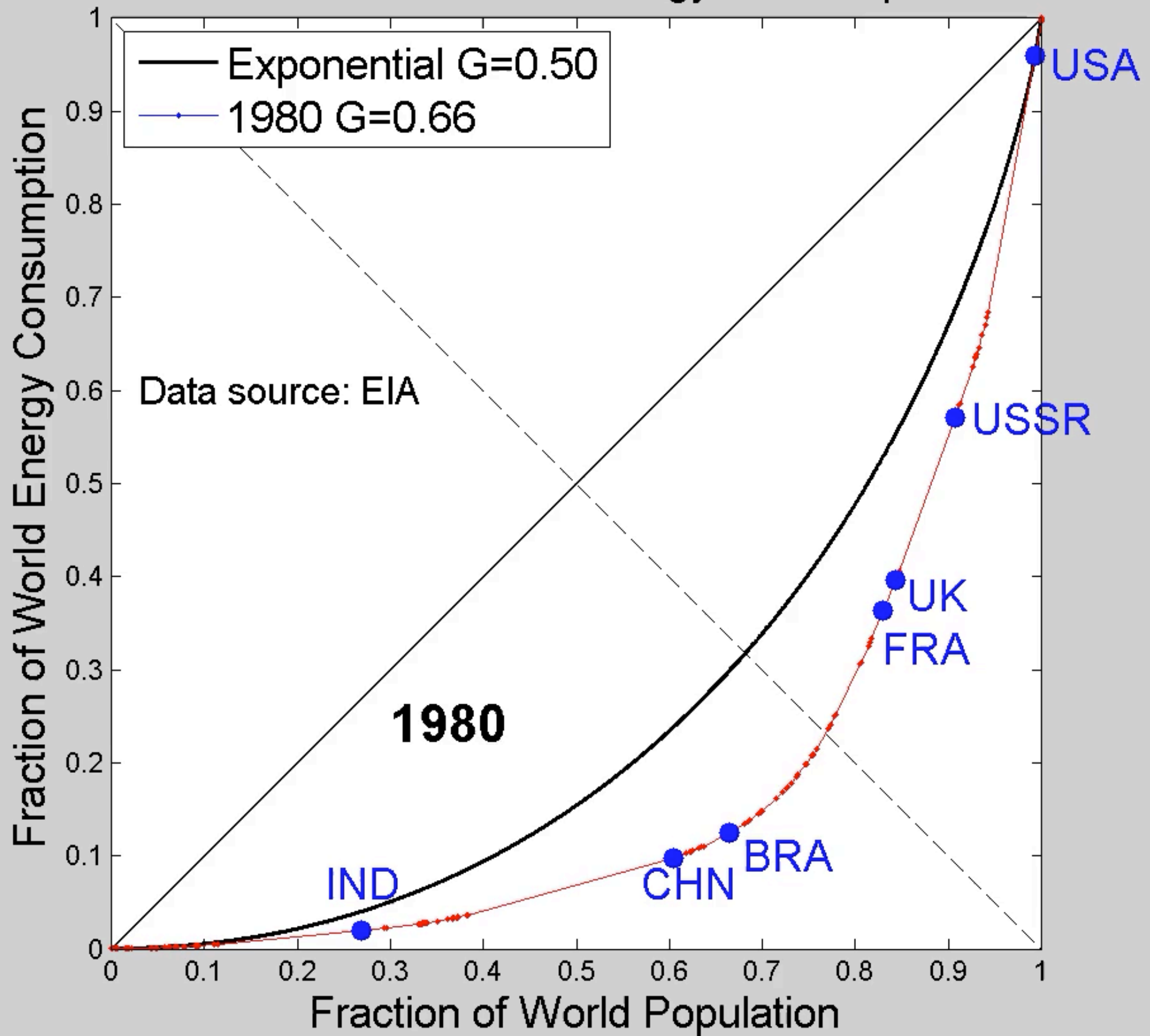


Global distribution of energy consumption per person is roughly **exponential**.

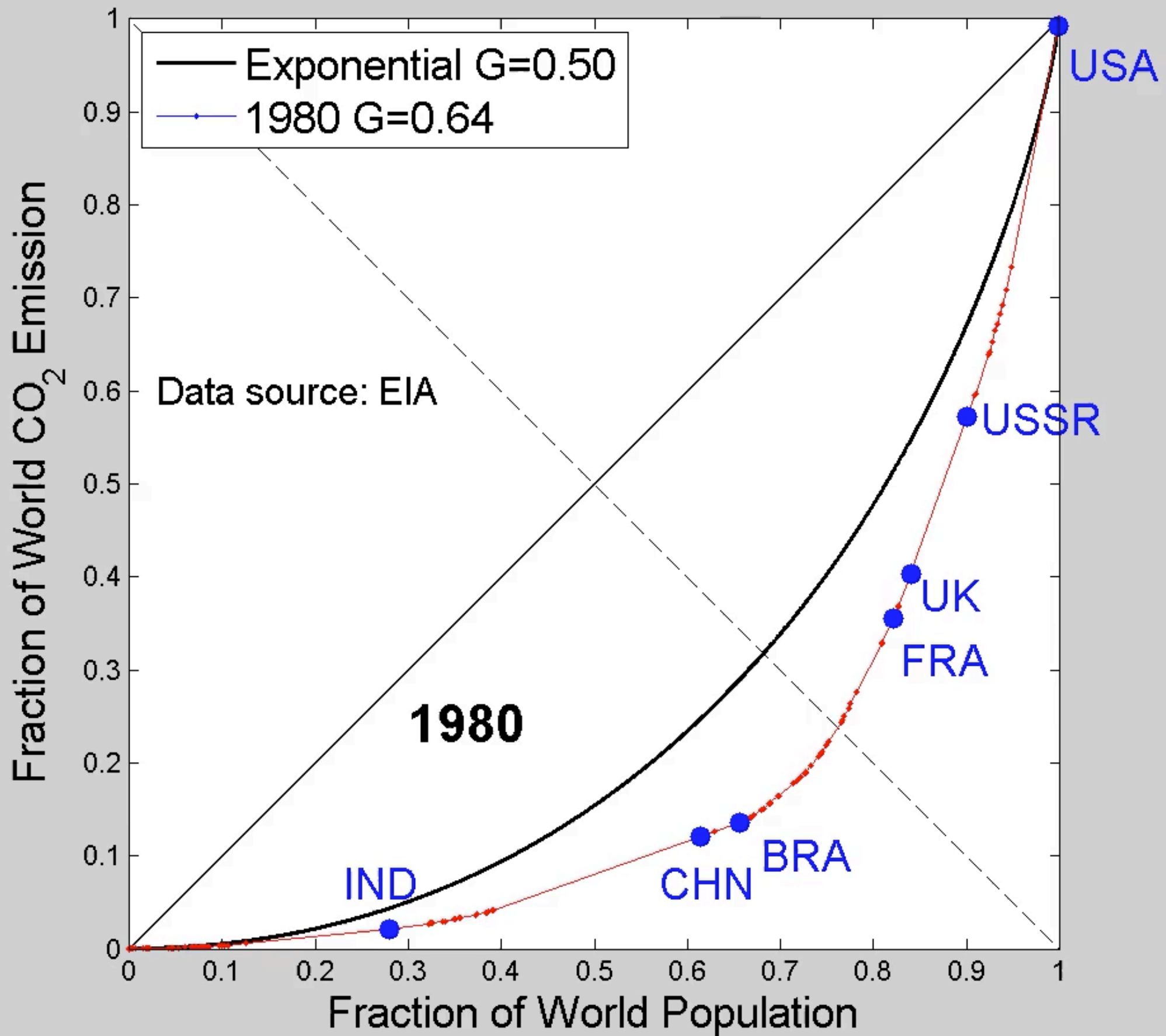
Division of a **limited resource** + **entropy maximization** produce **exponential distribution**.

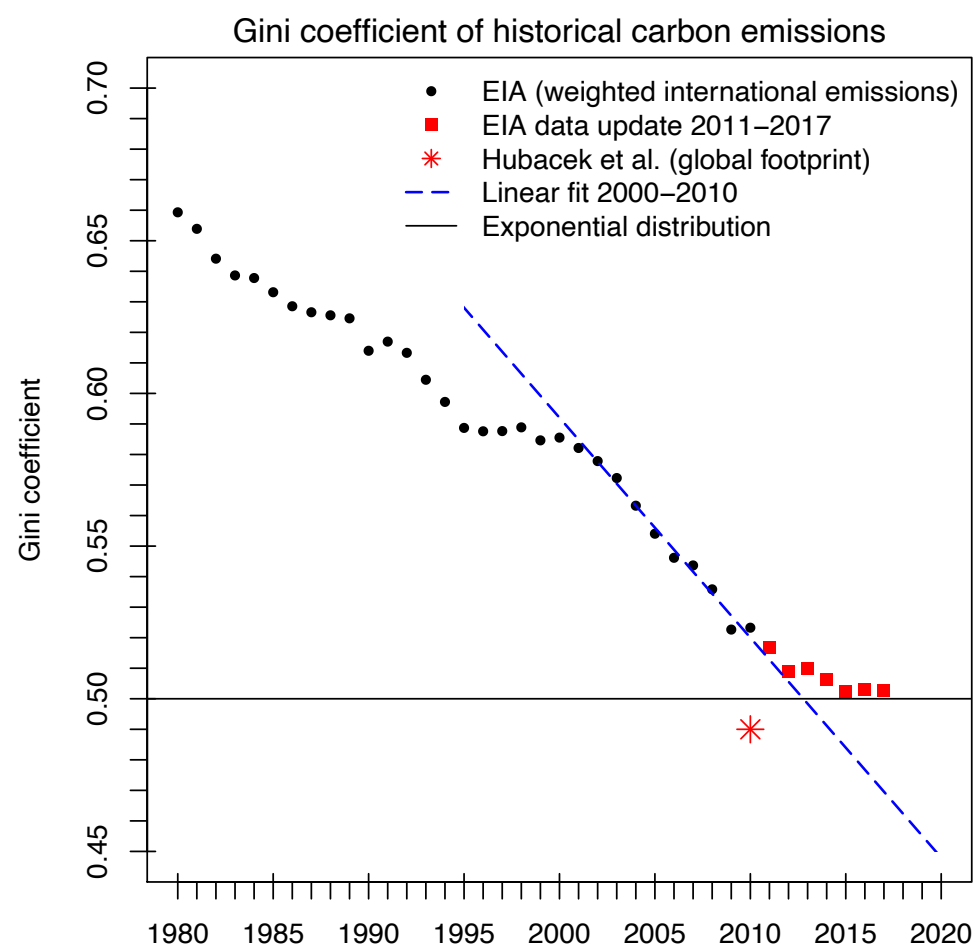
Physiological energy consumption of a human at rest is about **100 W**

Lorenz Plot of Global Energy Consumption

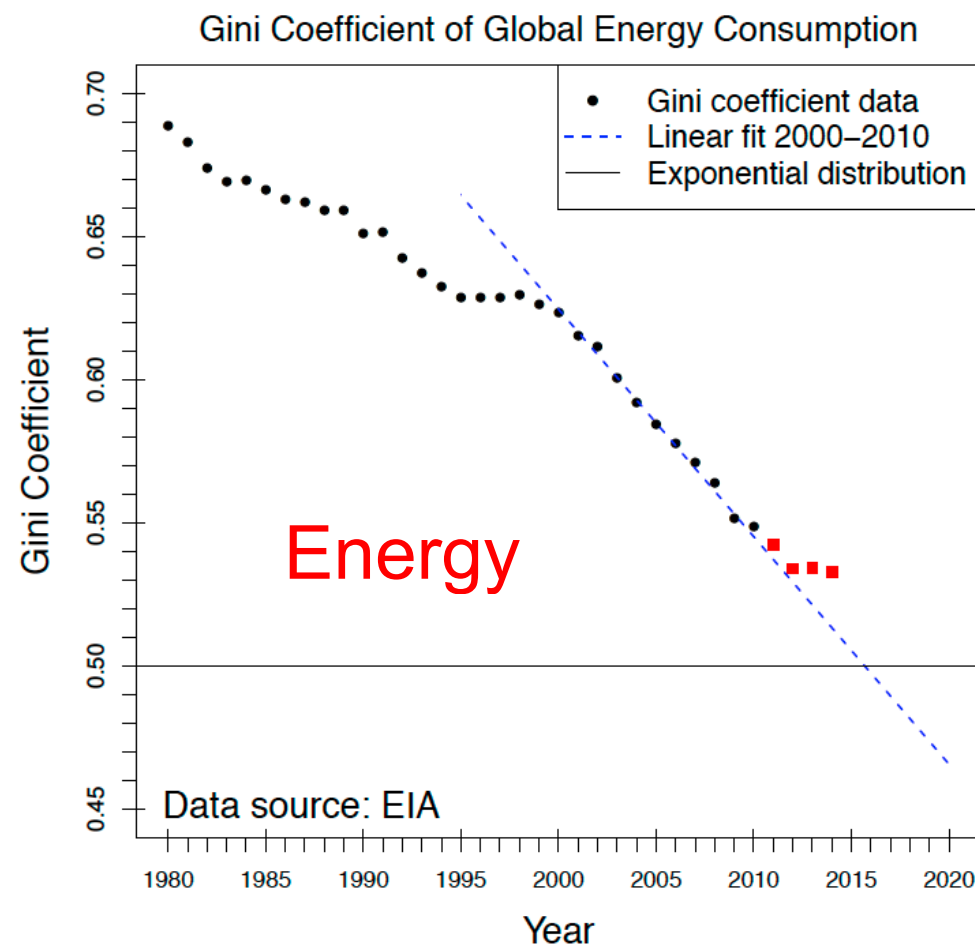
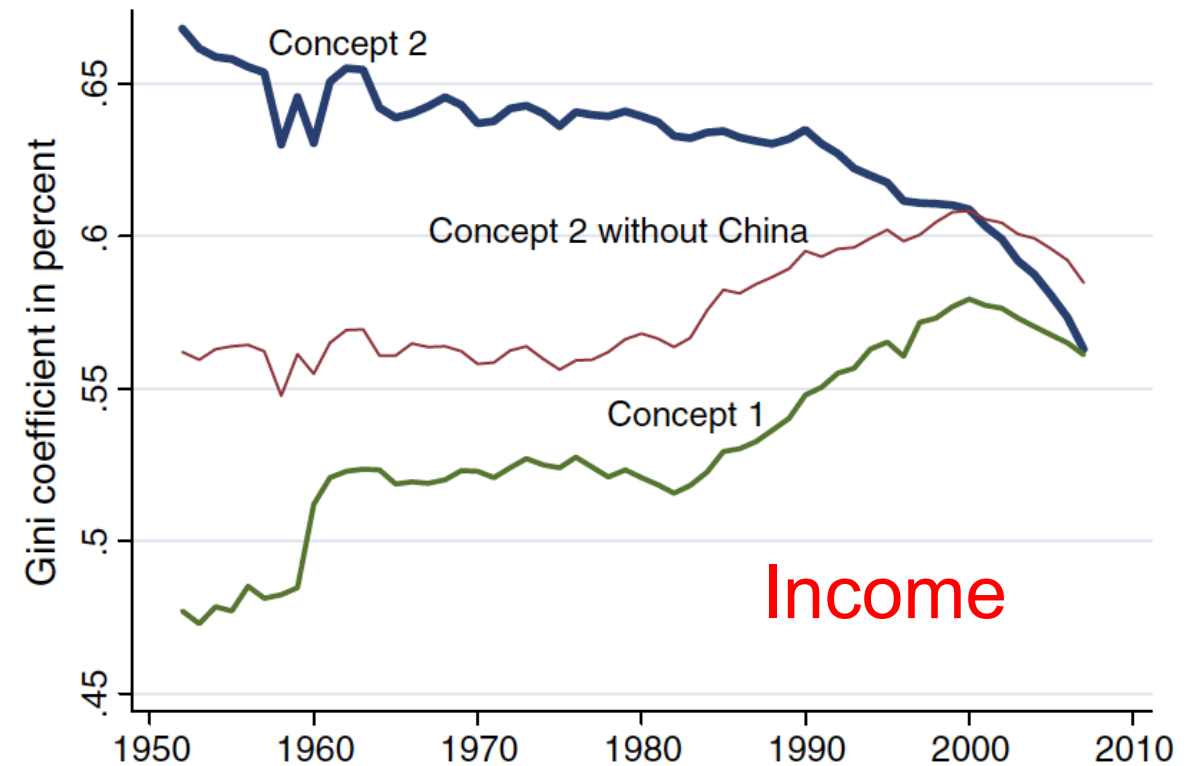


Lorenz Plot of Global CO₂ Emission

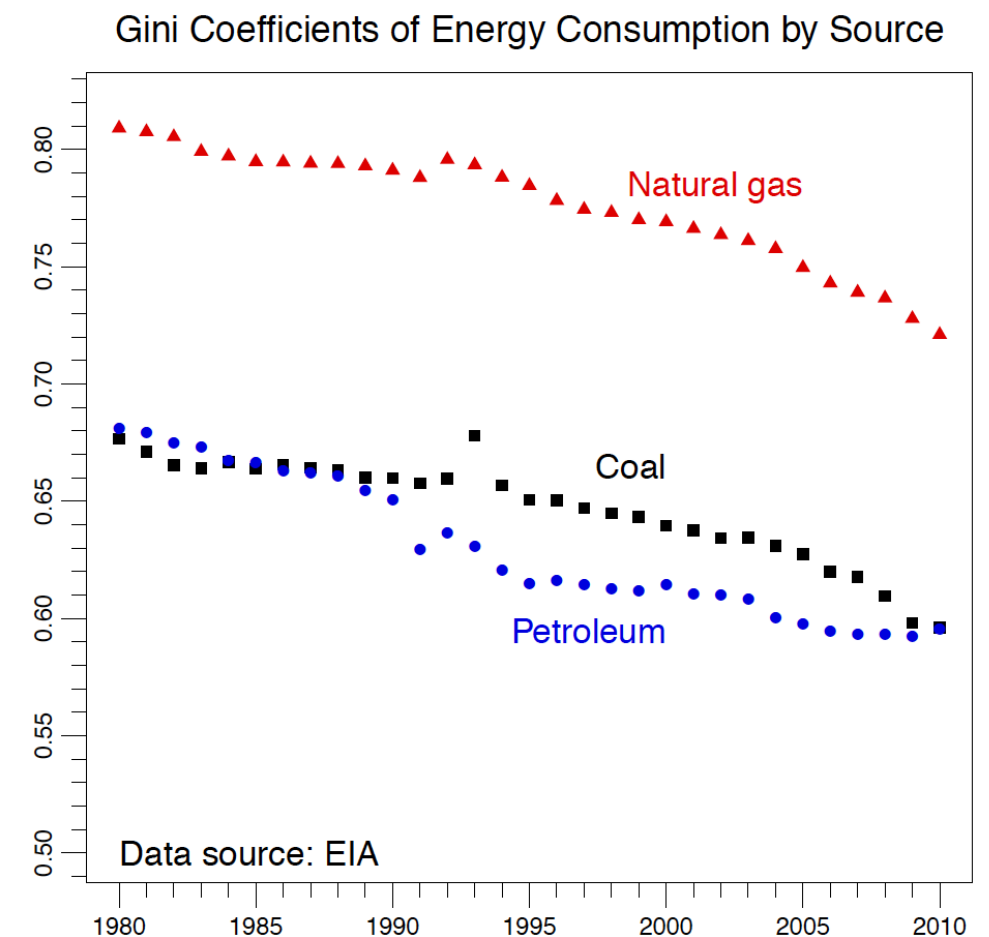




B. Milanovic, *J Econ Inequal* **10**, 1 (2012)



Money
=
Energy
=
Carbon

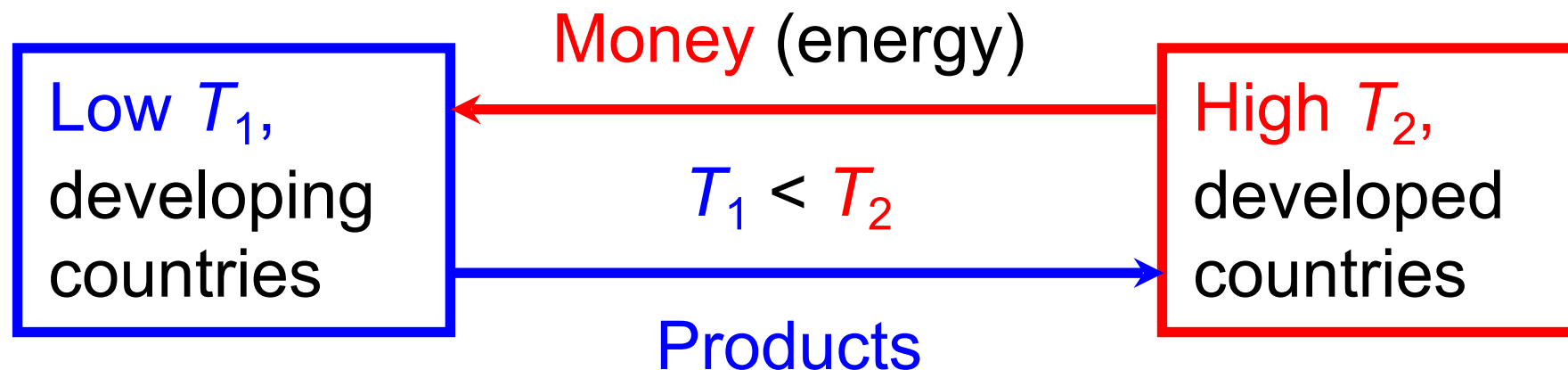


Conclusions

- The probability **distribution of money** is **stable** and has an **equilibrium** only when a **boundary condition**, such as $m > 0$, is imposed.
- When **debt** is permitted, the distribution of money becomes **unstable**, unless some sort of a **limit on maximal debt** is imposed.
- **Income distribution** in the USA has a **two-class structure**: **exponential** (“thermal”) for the great **majority (97-99%) of population** and **power-law** (“superthermal”) for the **top 1-3% of population**.
- The **exponential part** of the distribution is **very stable** and does not change in time, except for a **slow increase of temperature T** (the average income).
- The **power-law tail** is **not universal** and was increasing significantly for the last 20 years. It peaked and crashed in **2000** and **2007** with the **speculative bubbles** in financial markets.
- The global distribution of **energy consumption** per person is **highly unequal** and **roughly exponential**. This inequality is important in dealing with the global energy problems.
- All papers at <http://physics.umd.edu/~yakovenk/econophysics/>

Thermal machine in the world economy

In general, different countries have different temperatures T , which makes possible to construct a thermal machine:



Prices are commensurate with the income temperature T (the average income) in a country.

Products can be manufactured in a low-temperature country at a low price T_1 and sold to a high-temperature country at a high price T_2 .

The temperature difference $T_2 - T_1$ is the profit of an intermediary.

Money (energy) flows from high T_2 to low T_1 (the 2nd law of thermodynamics – entropy always increases) $\Leftrightarrow$ Trade deficit

In full equilibrium, $T_2 = T_1 \Leftrightarrow$ No profit $\Leftrightarrow$ "Thermal death" of economy